KNOTS AND CONTACT GEOMETRY

Plan of talks

- basic definitions

Knots and Contact Geometry

Basic definitions and examples

M^3 - oriented 3-manifold

def: a contact structure ξ on M is a completely non-integrable planefield on M

ξ is not tangent to any open surface

$\Updownarrow$

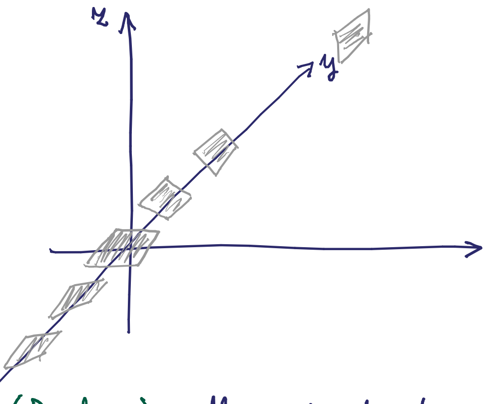
$\alpha \wedge d\alpha \neq 0$

$>$ we will require

locally ξ can be given as $\ker \alpha$ where α is a (local) 1-form

e.g.: $\mathbb{R}^3$ $\alpha_{st} = dz - y\,dx$ $\quad \xi_{st} = \ker \alpha_{st} = \{\partial y, y\partial z + \partial x\}$ $\quad d\alpha_{st} = -dx \wedge dy$

$\alpha_{st} \wedge d\alpha_{st} = -dz \wedge dx \wedge dy = dx \wedge dy \wedge dz$



standard contact structure

Thm (Darboux): all contact structures are locally diffeomorphic to $(\mathbb{R}^3, \xi_{st})$

moreover

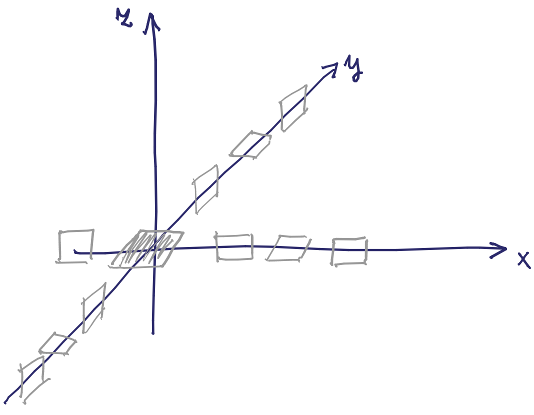
Thm (contact neighborhood Theorem): If ξ_0 and ξ_1 agree along a compact submanifold $L \subset M$. Then they are isotopic rel L in a neighborhood $N(L)$ of L

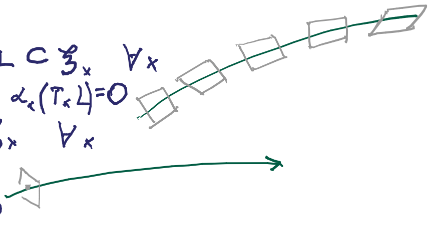
Thm (Gray's Theorem): homotopy of contact structures = isotopy of contact structures

e.g.: $\mathbb{R}^3$, $\alpha_{sym} = dz - y\,dx - x\,dy = dz + r^2 d\vartheta$ $\quad \xi_{sym} = \ker \alpha_{sym} = \{\partial r, r^2\partial z - \partial\vartheta\}$

HW1 find a diffeomorphism of $\mathbb{R}^3$ that takes ξ_{st} to ξ_{sym}

e.g.: $\mathbb{R}^3$, $\alpha_{OT} = \cos r\, dz + r \sin r\, d\vartheta$ $\quad \xi_{OT} = \ker \alpha_{OT}$





def: a knot $L \hookrightarrow (M,\xi)$ is a Legendrian knot if $T_xL \subset \xi_x \quad \forall x$

$\alpha_x(T_xL)=0$

$T \hookrightarrow (M,\xi)$ is a transverse knot if $T_xT \pitchfork \xi_x \quad \forall x$

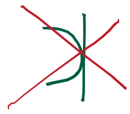
$\Leftrightarrow \alpha(TT) \neq 0$

Legendrian knots in the standard contact structure

$L \hookrightarrow (\mathbb{R}^3,\xi)$ Legendrian $\Leftrightarrow$ $dz - y\,dx = 0$ along L $\Leftrightarrow$ " $y = \frac{dz}{dx}$ " along L

front projection: projection to the (x,z)-plane

- $\frac{dz}{dx} = y \neq \pm\infty$: the front projection does not have vertical tangencies

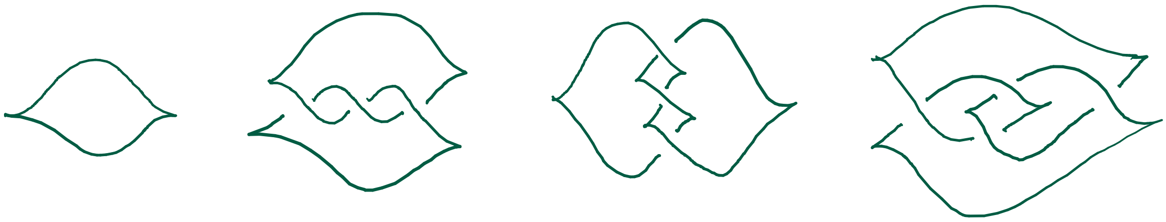


- instead cusps with well-defined tangent & immersion otherwise

moreover any planecurve with the above 2 properties is the projection of a Legendrian knot (y can be recovered as $\frac{dz}{dx}$)

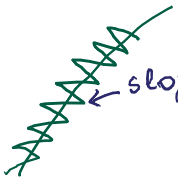
- X = X the slope of the overcrossing is smaller

e.g.:



Thm: Any knot can be C^0-approximated by a Legendrian knot.

proof: enough locally thus we can work in $(\mathbb{R}^3,\xi_{st})$

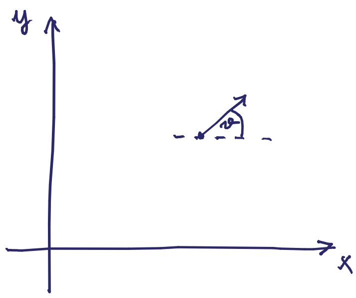


$\square$

Rmk: in $(\mathbb{R}^3,\xi_{std})$ there is a more efficient way of getting isotopic Legendrian knots to a given smooth knot:

Legendrian knots in real life

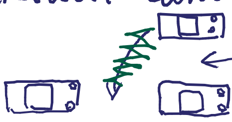
configuration space of state / front wheel of a car



- $(x,y,\vartheta) \in \mathbb{R}^2 \times S^1 = M$
- state goes where the front wheel points : $\frac{dy}{dx} = tg\,\vartheta$

$\xi := \ker(\cos\vartheta\, dy + \sin\vartheta\, dx)$ Legendrian curves $\leftrightarrow$ motion of state

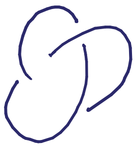
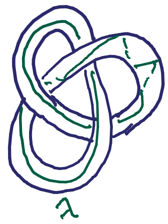
you can allways parallelpark your car



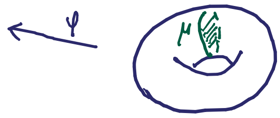
Application to topology

Kronheimer – Mrowka: nontrivial knots have Property P

Def: Dehn surgery on $K \subset S^3$:

 • remove $N(K)$: 

• glue back $S^1 \times D^2$ sending μ to λ



$\rightsquigarrow S^3_\lambda(K)$

K has Property P if non-trivial surgery yields nontrivial fundamental group

$$\lambda \neq \mu \Rightarrow \pi_1(S^3_\lambda(K)) \neq 1$$

Ozsváth - Szabó: the unknot, trefoil & figure eight knots are determined by their surgeries:

L =   or  then $S^3_p(K) = S^3_p(L)\ \forall p \Rightarrow K = L$

Giroux & Goodman (Harer's Conjecture): All fibered knots in S^3 are related by Hopf plumbing:

$K = \partial \Sigma$

$\longleftrightarrow$

$K' = \partial \Sigma'$

Classification of Legendrian knots

L_0, L_1 Legendrian knots are

- Legendrian isotopic if $\exists\, L_t$ $t \in [0,1]$ continous family of Legendrian knots
$\Updownarrow$
- ambient contact isotopic if $\exists\, \phi_t : M \to M$ 1-parameter family of contactomorphisms s.t. $\phi_0 = id$ & $\phi_1(L_0) = L_1$

Thm: Legendrian isotopy $\Leftrightarrow$ ambient contact isotopy

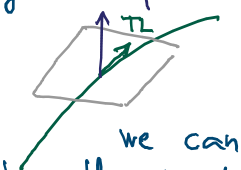
proof: $\exists$ Family of diffeomorphisms $\phi_t : M \to M$ $\quad \phi_t(L_0) = L_t$, $\phi_t^*(\xi|_{L_t}) = \xi|_{L_0}$

let $\xi_t = \phi_t^*(\xi) \xRightarrow[\text{Thm}]{\text{relative Gray}} \exists\, \psi_t$ diffeomorphisms $\psi_t^*(\xi_t) = \xi$ & $\psi_t|_{L_0} = id$

$f_t := \phi_t \circ \psi_t$ then $f_t^*(\xi_t) = \xi$ & $f_t(L_0) = \phi_t(L_0) = L_t$ $\square$

in (S^3, ξ_{st}) there is a third sense of classification equivalent to the above:

def: Thurston-Bennequin framing



$\Rightarrow$ for two Legendrian knots we can arrange these framings to be sent to each other, thus by the contact neighborhood thm any two Legendrian knots have contactomorphic neighborhoods $N(L)$

a modell for the standard Legendrian neighborhood:

$N_0 = D^2 \times S^1_\vartheta$, $\quad D^2 = \{(x,y) : x^2+y^2 \leq 1\}$ $\qquad \xi_0 = \ker(\underbrace{\cos\vartheta\, dx - \sin\vartheta\, dy}_{\alpha_0})$ $\qquad L_0 = 0 \times S^1$

$d\alpha_0 = -\sin\vartheta\, dx \wedge d\vartheta + \cos\vartheta\, dy \wedge d\vartheta$

$\alpha_0 \wedge d\alpha_0 = \cos^2\vartheta\, d\vartheta \wedge dx \wedge dy - \sin^2\vartheta\, dy \wedge dx \wedge d\vartheta$
$= (\cos^2\vartheta + \sin^2\vartheta)\, dx \wedge dy \wedge d\vartheta > 0$:)

Thm: In $(\mathbb{R}^3, \xi_{st})$ (or (S^3, ξ_{st})) L_0 is Legendrian isotopic to L_1 if their compliments $S^3 - N(L_0)$ and $S^3 - N(L_1)$ are contactomorphic.

proof: $S^3 - N(L_0) \to S^3 - N(L_1)$ contactomorphism given by Thm

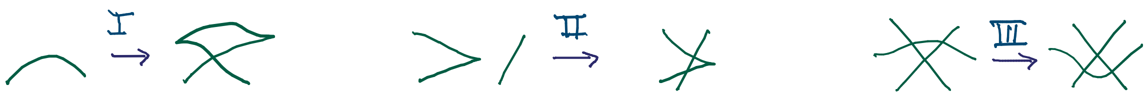
$N(L_0) \to N(L_1)$ contactomorphism coming from being standard contact neighborhoods

$\rightsquigarrow \varphi : S^3 \to S^3$ contactomorphism that sends L_0 to L_1

Thm (Eliashberg): The set of contactomorphisms of (S^3, ξ_{st}) is contractible.

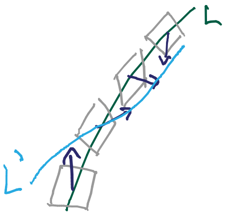
$\to$ connected: $\exists\, \varphi_t : S^3 \to S^3$ $\varphi_0 = id$ $\varphi_1 = \varphi$ $\square$

Thm: Two front diagrams represent Legendrian isotopic Legendrian knots iff they are related by regular homotopy & a sequence of the following moves



Classical invariants of Legendrian knots

- smooth knot type (Leg. isotopy ⇒ smooth isotopy)
- Thurston–Bennequin number: measures the twisting of ξ along L



if L is nullhomologous $tb(L) = lk(L, L')$

L is nullhomologous with Seifert surface Σ

- rotation number: obstruction to extend the trivialisation of ξ given by TL to Σ

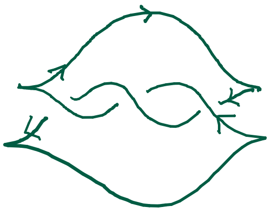
$$rot(L) = \langle e(\xi, TL), [\Sigma] \rangle$$

How to compute: trivialise

$$\begin{array}{ccc} \xi|_\Sigma \cong \Sigma \times \mathbb{R}^2 & \supset L \times \mathbb{R}^2 & \\ \downarrow & \downarrow \quad \uparrow TL & \rightsquigarrow \; S^1 \xrightarrow{f} \mathbb{R}^2 \setminus \{0\} \qquad rot(L) = \deg f \\ \Sigma & \supset L & \end{array}$$

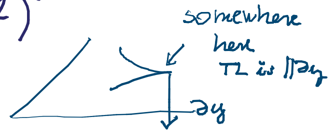
Computation of the classical invariants in the front projection

rotation number:

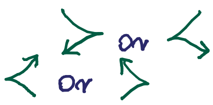


0

∂_y trivialises ξ on any Σ ⇒ need to compute winding of TL w.r.t ∂_y on ξ $= \frac{1}{2}$(signed number of counts where TL is vertical)



+ if counterclockwise
- if clockwise



$$rot(L) = \frac{1}{2}\left(d(L) - u(L)\right)$$

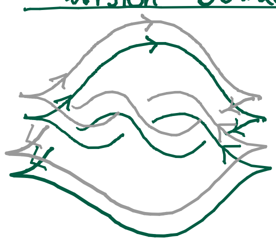
upwards cusps

downwards cusps

(counted both $\pm\partial_y$)

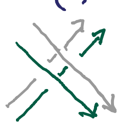
Thurston–Bennequin number

$\partial_z \pitchfork \xi$, push L along ∂_z to obtain L'



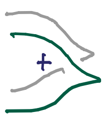
$\uparrow \partial_z$

$lk(L, L') = \frac{1}{2}$(signed # of crossings between L & L')

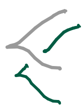


2 (sign # of crossing)

$$\Rightarrow lk(L, L') = writhe(L) - \frac{1}{2}c(L)$$



+1

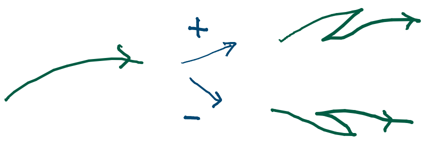


+1

Thm (Bennequin bound) In (S^3, ξ_{st}) $\quad tb(L) + |rot(L)| \leq -\chi(\Sigma)$

we prove a more general version of this in the next lecture

Stabilisation: change L locally

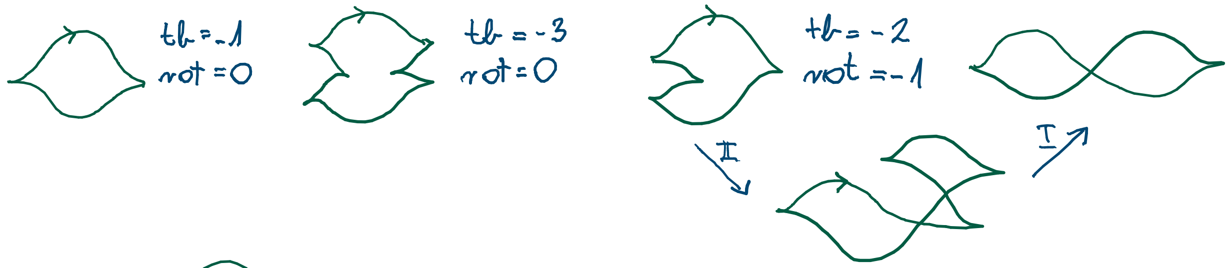


$$tb(L^\pm) = tb(L) - 1$$

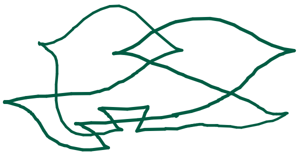
$$rot(L^\pm) = rot(L) \pm 1$$

(HW) prove that stabilisation is a well-defined operation!

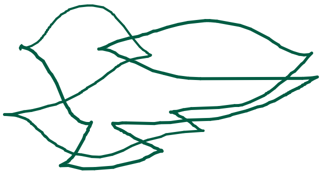
e.g.:



(HW) Show



and



are Legendrian isotopic.

Classification of Legendrian knots

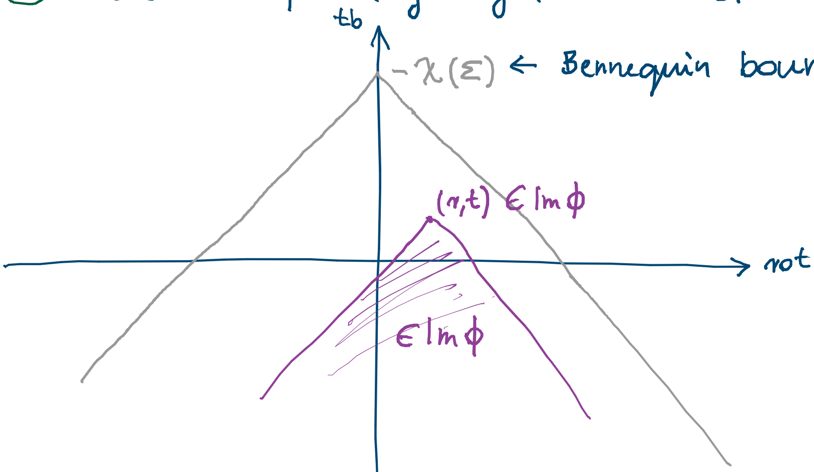
K smooth knot type : $\mathcal{L}(K) = $ Legendrian knots L in $(\mathbb{R}^3, \xi_{st})$ smoothly isot. to K / Leg isot

$$\phi : \mathcal{L}(K) \longrightarrow \mathbb{Z} \times \mathbb{Z}$$
$$L \longmapsto (rot(L), tb(L))$$

classifying Legendrian knots → geography: determine $\operatorname{Im}\phi$
→ botany: for $(r,t) \in \operatorname{Im}\phi$ determine $\phi^{-1}(r,t)$

def: K is Legendrian simple if ϕ is injective : Legendrian knots representing K are classified by (rot, tb).

(HW) Prove that for any Legendrian knot tb + rot is odd



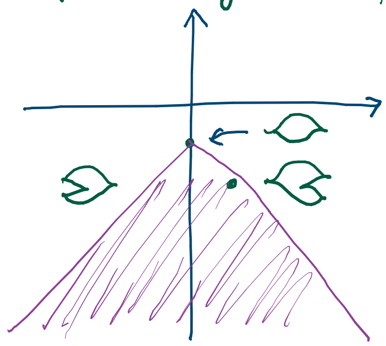
← Bennequin bound ⇒ $\operatorname{Im}\phi$ is contained in the triangle

symmetric to this : $(x,y,z) \xmapsto{f} (-x,y,-z)$ co-actomorphism isotopic to id

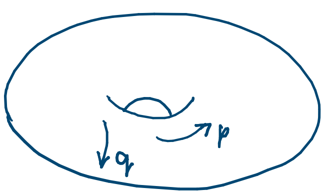
$tb(f(L)) = tb(L)$
$rot(f(L)) = -rot(L)$

Classification results:

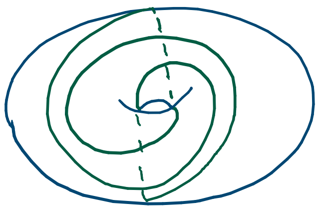
- **Thm** (Eliashberg - Fraser, 1995): the unknot is Legendrian simple: (will prove this later)



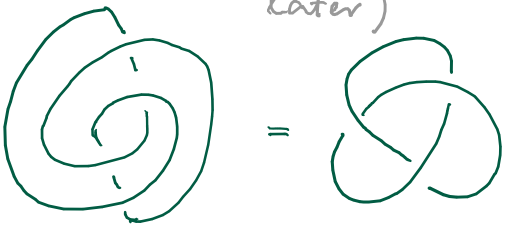
- **Thm** (Etnyre - Honda, 2001): torus knots are Legendrian simple (will prove this later)



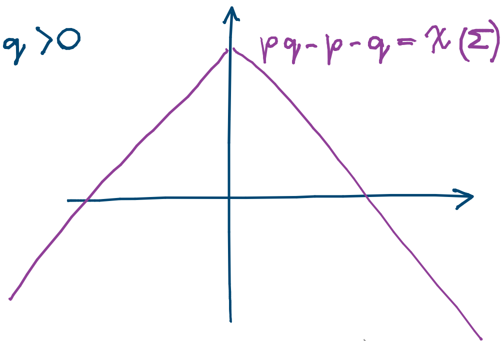
(p,q)-torus knot
$(p,q)=1 \quad q>0$

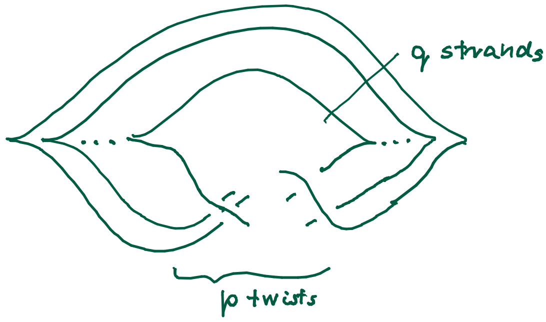


(2,3) torus knot

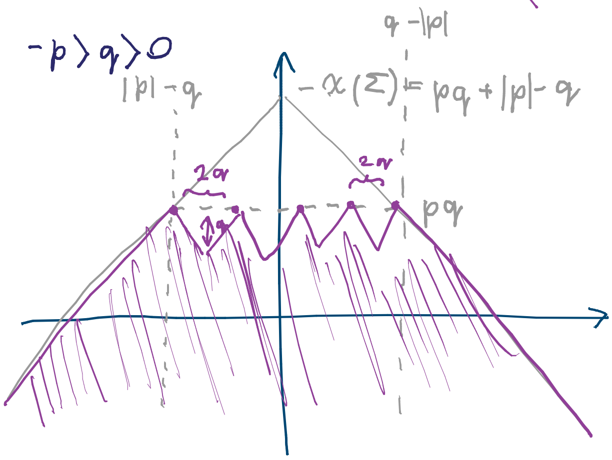


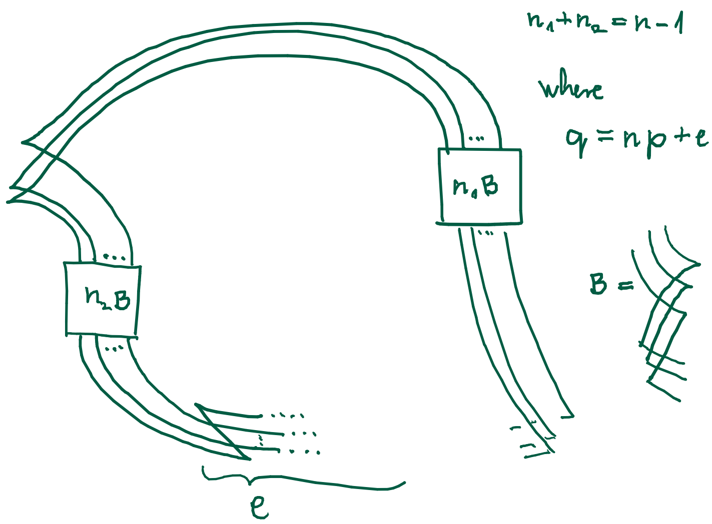
- if $p,q>0$





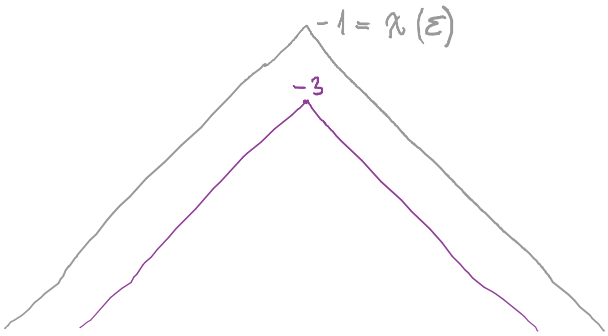
- if $-p>q>0$

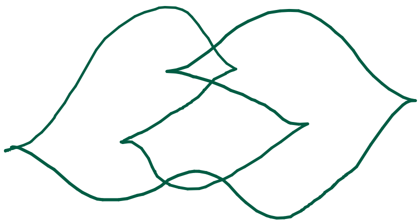




(arbitrarily many peaks & valleys)

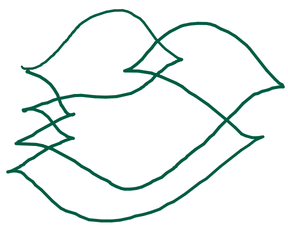
Thm (Etnyre - Honda) figure eight knot is Legendrian simple





but:

<u>Thm</u> (Chekanov):



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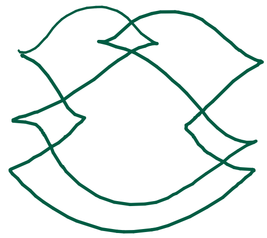


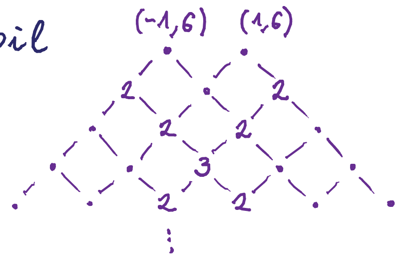
are not Legendrian isotopic

tb = -1 rot = 0

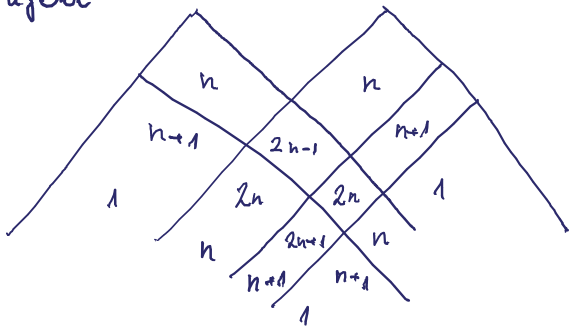
for distinguishing them: contact homology (Chekanov, Eliashberg - Hofer)
Heegaard Floer homology (will talk about this later)

more classifications:

- Etnyre - Honda 2005: (2,3) cable of the trefoil



- Etnyre - LaFountain - Tosun: (p,q) - cable of trefoil

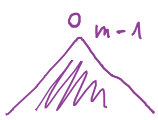


- Etnyre - Ng - V based on results of Ozsváth - Stipsicz: twist knots

K_m

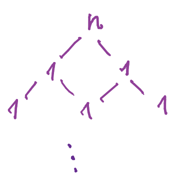
$m \geq 0 \Rightarrow K_m$ Legendrian simple

m odd

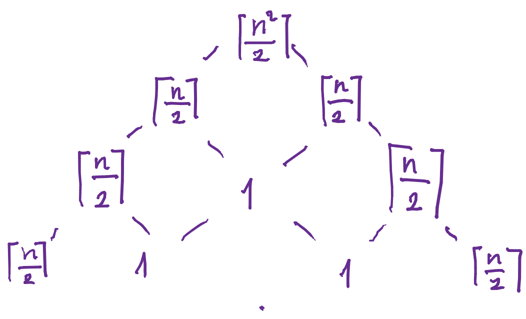


m even

$m < 0$ $\quad m = -2n-1$



m odd



Structural results